Analysis of Set-theoretic Unknown Input Observer and Interval Observer in Robust Fault Detection

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***Abstract*— This paper focuses on comparing fault detection (FD) performance of two robust FD approaches, namely the set-theoretic unknown input observer-based approach and the interval observer-based approach. The former is implemented using the set theory and the unknown input observer (UIO), where the set theory and the UIO are used to decouple the effect of unknown inputs in passive and active ways, respectively. The latter is based on the set theory and the Luenberger observer, which completely relies on the set-based passive decoupling to obtain FD robustness. The former is a new method recently proposed by the authors, while the latter is a well-known robust FD method. The objective of this paper is to analyze these two FD approaches in a systematic way and to compare their FD sensitivity based on mathematical analysis. Eventually, an explicit criterion used to choose which method is better for FD of a system is given by using invariant sets. At the end of this paper, an example is used to illustrate the obtained results.**

I. Introduction

There always exist disturbances and noises in a technical system. If we want to design an FD module to monitor the system operation, the chosen FD approach must be able to cope with these factors by achieving the so-called FD robustness. Otherwise, the FD approach will not be able to effectively detect faults in the system [5].

In general, we can obtain FD robustness by either a passive or active way. A well-known active approach is based on the UIO. Particularly, if a system can satisfy the design conditions of UIOs, we can design a UIO to actively decouple the effect of all its unknown inputs [1], [2]. In this case, the residuals generated by the UIO can be insensitive to all the unknown inputs but sensitive to faults. However, due to the restriction of the design conditions of UIOs, the application of this active decoupling approach is quite limited. In [9], [10], the authors proposed a novel design of unknown input observers to overcome this limitation, which is based on the set-theoretic unknown input observer (SUIO) that combines the active and passive approaches. Regarding the passive FD approach, we use the set theory to decouple the effect of unknown inputs and noises and the principle of the passive approaches can be found in [6], [7].

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The SUIO-based method can remove the effect of part of unknown inputs with the help of the active decoupling capability of UIO. In this way, the FD conservatism originated from unknown inputs can be reduced because the unknown inputs actively decoupled will not contribute to the size of generated sets. However, since the UIO is not able to decouple measurement noises, the effect of measurement noises is introduced to the state dynamics of UIO, which increases the FD conservatism. For the method based on the interval observer (IO), all the unknown inputs and measurement noises are passively decoupled and contribute to the size of generated sets. However, the effect of measurement noises on the FD conservatism is lower than that in the SUIO-based method. Since both methods have their own advantages and disadvantages, comparing them is the main point of this paper. The objective of this paper is to give mathematical results that can show the advantages and disadvantages of both approaches in robust FD.

The remainder of this paper is organized as follows. Section II introduces the FD principles of the two methods. Section III analyzes and compares their FD performance. Section IV illustrates the obtained results with a numerical example. The paper is concluded in Section V.

II. System Description

This section introduces the system model and the methods used to design an SUIO and an IO.

A. System Model

The linear discrete time-invariant plant is modeled as

$$x_{k+1} = Ax_k + Bu_k + E\omega_k, \tag{1a}$$

$$y_k = Cx_k + F\eta_k, \tag{1b}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $E \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{q\times n}$ and $F \in \mathbb{R}^{q\times s}$ are time-invariant matrices, k denotes the k-th discrete time, $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^q$ are the state and output vectors, $u_k \in \mathbb{R}^p$ and $\omega_k \in \mathbb{R}^r$ are known and unknown inputs (i.e., disturbances, modeling errors, linearizing errors, etc.), and $\eta_k \in \mathbb{R}^s$ represents the measurement noise vector.

Assumption 2.1: The pair (A, C) is detectable.

According to [1], [10], in the case that there does not exist a UIO able to actively decouple all unknown inputs included in ω_k (i.e., cannot satisfy the traditional design conditions of UIO), we can design an SUIO to overcome this limitation by only decoupling a part of unknown inputs. In order to design an SUIO proposed in [10], the unknown input vector ω_k should be divided into

$$\omega_k = \begin{bmatrix}\omega_{1,k}^T & \omega_{2,k}^T\end{bmatrix}^T, \tag{2}$$

where $\omega_{1,k} \in \mathbb{R}^{n_a}$, $\omega_{2,k} \in \mathbb{R}^{n_p}$, n_a is defined as the number of unknown inputs that the SUIO is designed to actively decouple, while n_p denotes the remaining number of unknown inputs and is computed by $n_p = r - n_a$.

Assumption 2.2: $\omega_{1,k}$, $\omega_{2,k}$, ω_k and η_k are bounded by

$$\begin{aligned} W_1 =&\{\omega_1 \in \mathbb{R}^{n_a} : |\omega_1 - \omega_1^c| \le \bar{\omega}_1\}, \\ W_2 =&\{\omega_2 \in \mathbb{R}^{n_p} : |\omega_2 - \omega_2^c| \le \bar{\omega}_2\}, \\ W =&\{\omega \in \mathbb{R}^{n} : |\omega - \omega^c| \le \bar{\omega}\}, \\ V =&\{\eta \in \mathbb{R}^{s} : |\eta - \eta^c| \le \bar{\eta}\}, \end{aligned}$$

where ω_1^c, $\bar{\omega}_1$, ω_2^c, $\bar{\omega}_2$, ω^c, $\bar{\omega}$, η^c and $\bar{\eta}$ are constant vectors.

Remark 1: The boundness assumption of $\omega_{1,k}$ is only made for the IO-based approach, while in the SUIO-based approach, since $\omega_{1,k}$ has been actively decoupled, it is not required that $\omega_{1,k}$ is bounded, which is an advantage of the SUIO-based approach over the IO-based approach.

Under Assumption 2.2, the sets W_1, W_2, W and V can be rewritten into zonotopic form as

$$\begin{aligned} W_1 =&\{\omega_1^c\} \oplus H_{\bar{\omega}_1}\mathbb{B}^{n_a},\ W_2 = \{\omega_2^c\} \oplus H_{\bar{\omega}_2}\mathbb{B}^{n_p}, \\ W =&\{\omega^c\} \oplus H_{\bar{\omega}}\mathbb{B}^{n},\ V = \{\eta^c\} \oplus H_{\bar{\eta}}\mathbb{B}^{s}, \end{aligned}$$

where $H_{\bar{\omega}_1}$, $H_{\bar{\omega}_2}$, $H_{\bar{\omega}}$ and $H_{\bar{\eta}}$ are diagonal matrices, whose diagonal elements are composed of $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{\omega}$ and $\bar{\eta}$, respectively, and $\oplus$ denotes the Minkowski sum.

Taking (2) into account, the matrix E is decomposed into

$$E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \tag{5}$$

where $E_1 \in \mathbb{R}^{n \times n_a}$ and $E_2 \in \mathbb{R}^{n \times n_p}$.

B. SUIO-based Fault Detection

In order to actively decouple $\omega_{1,k}$, we design an SUIO as

$$z_{k+1} = N z_k + T u_k + K y_k, \tag{6a}$$
$$\hat{x}_k^{suio} = M z_k + H y_k, \tag{6b}$$
$$\hat{y}_k^{suio} = C \hat{x}_k^{suio}, \tag{6c}$$

where $z_k \in \mathbb{R}^n$, $\hat{x}_k^{suio} \in \mathbb{R}^n$ and $\hat{y}_k^{suio} \in \mathbb{R}^q$ are the state vector, the estimated state and output vectors of the SUIO, $N \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{n\times p}$, $K \in \mathbb{R}^{n\times q}$, $M \in \mathbb{R}^{n\times n}$ and $H \in \mathbb{R}^{n\times q}$. According to [10], the parametric matrices of (6) as an SUIO should satisfy

$$E_1 - HCE_1 = \mathbf{0}, \tag{7a}$$
$$B - MT - HCB = \mathbf{0}, \tag{7b}$$
$$(A - HCA - MK_1C)M - MN = \mathbf{0}, \tag{7c}$$
$$(A - HCA - MK_1C)H - MK_2 = \mathbf{0}, \tag{7d}$$

with $K = K_1 + K_2$. Furthermore, with (1), (6) and (7), the state-estimation error is defined as

$$e_k = x_k - \hat{x}_k^{suio} \tag{8}$$

and the dynamics of the state-estimation error of the SUIO can be further obtained as

$$\begin{aligned} e_{k+1} =&(A - HCA - MK_1C)e_k + (E_2 - HCE_2)\omega_{2,k} \\ &- HF\eta_{k+1} - MK_1F\eta_k, \end{aligned} \tag{9}$$

where it can be seen that the effect of $\omega_{1,k}$ has been actively decoupled by the SUIO while $\omega_{2,k}$ and η_k still affect e_k.

Remark 2: According to [10], under the conditions (7), it is almost impossible to design parametric matrices to further satisfy $HF = \mathbf{0}$ or $MK_1F = \mathbf{0}$. Thus, we only consider the case $HF \neq 0$ and $MK_1F \neq 0$ here (see (9)).

It is assumed that, the initial set of the state-estimation error is denoted as E_0^r. Then, we can use the set theory to passively decouple $\omega_{2,k}$. Based on (9), the set-based dynamics of the state-estimation error can be obtained as

$$\begin{aligned} E_{k+1}^r =&(A - HCA - MK_1C)E_k^r \oplus (E_2 - HCE_2)W_2 \\ &\oplus HF(-V) \oplus MK_1F(-V). \end{aligned} \tag{10}$$

Remark 3: If $e_0 \in E_0^r$ holds, $e_k \in E_k^r$ for all $k \ge 0$ can be guaranteed, as long as no faults occur in the system.

For the FD purpose, the residual vector can be defined as

$$r_k^{suio} = y_k - \hat{y}_k^{suio} = Ce_k + F\eta_k. \tag{11}$$

Using (11), we can further construct the set of r_k^{suio} as

$$R_k^{suio} = CE_k^r \oplus FV. \tag{12}$$

Using zonotope[1] operations, the equations (10) and (12) can be equivalently transformed into

$$\begin{aligned} e_{k+1}^c =&(A - HCA - MK_1C)e_k^c + (E_2 - HCE_2)\omega_2^c \\ &- (H + MK_1)F\eta^c, \end{aligned} \tag{13a}$$
$$\begin{aligned} H_{k+1}^e =&[(A - HCA - MK_1C)H_k^e \ \ (E_2 - HCE_2)H_{\bar{\omega}_2} \\ &- HFH_{\bar{\eta}} \ \ - MK_1FH_{\bar{\eta}}], \end{aligned} \tag{13b}$$
$$r_k^{suio,c} = Ce_k^c + F\eta^c, \tag{13c}$$
$$H_k^{suio} = [CH_k^e \ \ FH_{\bar{\eta}}], \tag{13d}$$

where e_k^c, $r_k^{suio,c}$, H_k^e and H_k^{suio} are the centers and segment matrices of E_k^r and R_k^{suio}, respectively.

According to the above analysis, the FD criterion of the SUIO-based approach can be given as

$$r_k^{suio} \in R_k^{suio}, \tag{14}$$

where both r_k^{suio} and R_k^{suio} are obtained in real-time.

However, in order to compare the FD sensitivity of the SUIO-based approach and the IO-based approach. The FD criterion (14) is equivalently transformed into

$$\mathbf{0} \in \bar{R}_k^{suio} = R_k^{suio} \oplus \{-r_k^{suio}\}, \tag{15}$$

Furthermore, with (9), (10), (11), (12), (13) and (14), $\bar{R}_k^{suio}$ can be rewritten into the zonotopic form

$$\bar{R}_k^{suio} = \bar{r}_k^{suio} \oplus \bar{H}_k^{suio}\mathbb{B}^{s_k}, \tag{16}$$

[1]A zonotope Z is defined as $Z = g \oplus H\mathbb{B}^r$, where g and H are its center and segment matrix, respectively, and $\mathbb{B}^r$ is a box composed of r unitary intervals. Given $X_1 = g_1 \oplus H_1\mathbb{B}^{r_1}$ and $X_2 = g_2 \oplus H_2\mathbb{B}^{r_2}$, $X_1 \oplus X_2 = \{g_1 + g_2\} \oplus [H_1 \ \ H_2]\mathbb{B}^{r_1+r_2}$. Given $X = g \oplus H\mathbb{B}^r$ and a suitable matrix K, $KX = Kg \oplus KH\mathbb{B}^r$.

where s_k denotes the order of $\bar{R}_k^{suio}$ at time instant k and

$$\begin{aligned}
\bar{r}_k^{suio} =& C\tilde{e}_k + F\tilde{\eta}_k, &(17a)\\
\bar{H}_k^{suio} =& H_k^{suio} = [CH_k^e \;\; FH_{\bar{\eta}}], &(17b)\\
\tilde{e}_k =& e_k^c - e_k, &(17c)\\
\tilde{\omega}_{2,k-1} =& \omega_2^c - \omega_{2,k-1}, &(17d)\\
\tilde{\eta}_k =& \eta^c - \eta_k, &(17e)\\
\tilde{e}_k =& (A - HCA - MK_1C)\tilde{e}_{k-1} - HF\tilde{\eta}_k \\
&+ (E_2 - HCE_2)\tilde{\omega}_{2,k-1} - MK_1F\tilde{\eta}_{k-1}, &(17f)
\end{aligned}$$

which implies that if a violation of (15) is detected, the system has become faulty. Otherwise, it is assumed that the system is still in healthy operation.

C. IO-based Fault Detection

According to [6], [8], an IO with Luenberger structure can be designed to monitor the system (1) as

$$\begin{aligned}
\hat{X}_{k+1} =& (A - LC)\hat{X}_k \oplus \{Bu_k\} \oplus \{Ly_k\} \\
&\oplus (-LF)V \oplus EW, &(18a)\\
\hat{Y}_k =& C\hat{X}_k \oplus FV, &(18b)
\end{aligned}$$

where $\hat{X}_k$ and $\hat{Y}_k$ are estimated state and output sets, and L is the observer gain chosen to ensure the observer stability. Using zonotopes, (18) can be equivalently transformed into the center-segment matrix form

$$\hat{x}_{k+1}^c = (A - LC)\hat{x}_k^c + Bu_k + Ly_k - LF\eta^c + Ew^c, \qquad (19a)$$

$$\begin{aligned}
\hat{H}_{k+1}^x =& [(A - LC)\hat{H}_k^x \quad -LFH_{\bar{\eta}} \quad EH_{\bar{\omega}}], &(19b)\\
\hat{y}_k^c =& C\hat{x}_k^c + F\eta^c, &(19c)\\
\hat{H}_k^y =& [C\hat{H}_k^x \quad FH_{\bar{\eta}}], &(19d)
\end{aligned}$$

where $\hat{x}_{k+1}^c$ and $\hat{H}_{k+1}^x$, and $\hat{y}_k^c$ and $\hat{H}_k^y$ are the centers and segment matrices of $\hat{X}_{k+1}$ and $\hat{Y}_k$, respectively.

Assumption 2.3: The initial state satisfies $x_0 \in \hat{X}_0$.

Since the IO is based on the healthy model of the system, the estimated output set can always bound the outputs, i.e.,

$$y_k \in \hat{Y}_k, \qquad (20)$$

which is the FD criterion of the IO-based approach. In order to help the comparison with the SUIO-based method, the IO-based FD criterion can be equivalently transformed into

$$\begin{aligned}
\mathbf{0} \in \bar{R}_k^{io} =& \hat{Y}_k \oplus \{-y_k\} \\
=& C\hat{X}_k \oplus \{-Cx_k\} \oplus FV \oplus \{-F\eta_k\}. &(21)
\end{aligned}$$

Thus, if (21) is violated, the system has become faulty. Otherwise, it is assumed that the system is still healthy. The residual set in (21) can be rewritten into zonotopic form

$$\bar{R}_k^{io} = \bar{r}_k^{io} \oplus \bar{H}_k^{io}\mathbb{B}^{i_k}, \qquad (22)$$

where i_k denotes the order of $\bar{R}_k^{io}$ at time instant k and according to (1) and (19), we can have

$$\begin{aligned}
\bar{r}_k^{io} =& C\tilde{x}_k^c + F\tilde{\eta}_k, &(23a)\\
\bar{H}_k^{io} =& [C\hat{H}_k^x \;\; FH_{\bar{\eta}}], &(23b)\\
\tilde{x}_k^c =& \hat{x}_k^c - x_k, &(23c)\\
\tilde{\omega}_k =& \omega^c - \omega_k, &(23d)\\
\tilde{x}_{k+1}^c =& (A - LC)\tilde{x}_k^c - LF\tilde{\eta}_k + E\tilde{\omega}_k. &(23e)
\end{aligned}$$

III. Comparative Analysis

This section will compare the FD performance of the SUIO-based approach and the IO-based approach.

A. Residual Analysis

In order to compare the two approaches, we use the FD criteria (15) and (21), where both consist in testing whether the origin $\mathbf{0}$ is contained in the residual zonotopes estimated by the SUIO and the IO, respectively. This implies that we only need to consider the variation of the estimated residual zonotopes $\bar{R}_k^{suio}$ and $\bar{R}_k^{io}$ in (15) and (21) while the variation of the residual and output signals r_k^{suio} and y_k in (14) and (20) can be avoided. In this case, only one variable needs to be considered for comparison of the two approaches.

By analyzing (17d), (17e) and (23d), we can obtain

$$\begin{aligned}
\tilde{\omega}_{2,k} \in \bar{W}_2 =& \{\omega_2^c\} \oplus (-W_2), &(24a)\\
\tilde{\omega}_k \in \bar{W} =& \{\omega^c\} \oplus (-W), &(24b)\\
\tilde{\eta}_k \in \bar{V} =& \{\eta^c\} \oplus (-V), &(24c)
\end{aligned}$$

where $\bar{W}_2$, $\bar{W}$ and $\bar{V}$ are centered at the origin.

Moreover, by comparing (15), (16) and (17) with (21), (22) and (23), respectively, we can relate the two FD methods by comparing (13b), (17f), (19b) and (23e).

Substituting $\bar{W}_2$, $\bar{W}$ and $\bar{V}$ into (17f) and (23e) to replace $\tilde{\omega}_k^2$, $\tilde{\omega}_k$ and $\tilde{\eta}_k$, respectively, we can obtain

$$\begin{aligned}
\tilde{E}_{k+1} =& (A - HCA - MK_1C)\tilde{E}_k \oplus (E_2 - HCE_2)\bar{W}_2 \\
&\oplus (-HF\bar{V}) \oplus (-MK_1F\bar{V}), &(25a)\\
\tilde{X}_{k+1}^c =& (A - LC)\tilde{X}_k^c \oplus (-LF\bar{V}) \oplus E\bar{W}, &(25b)
\end{aligned}$$

where $\tilde{E}_k$ and $\tilde{X}_k^c$ are the sets of $\tilde{e}_k$ and $\tilde{x}_k^c$, respectively.

Remark 4: It can be observed that (25a) and (25b) are the set-based versions of (17f) and (23e), respectively. Thus, as long as $\tilde{e}_0 \in \tilde{E}_0$ and $\tilde{x}_0^c \in \tilde{X}_0^c$ hold, $\tilde{e}_k \in \tilde{E}_k$ and $\tilde{x}_k^c \in \tilde{X}_k^c$ will always hold for $k > 0$ if no faults occur.

By using zonotope operations given in Footnote 1, (25a) and (25b) can be further decomposed into the center-segment matrix form of zonotope:

$$\begin{aligned}
\tilde{e}_{k+1}^c =& (A - HCA - MK_1C)\tilde{e}_k^c, &(26a)\\
H_{k+1}^{\tilde{e}} =& H_{k+1}^e, &(26b)\\
\tilde{x}_{k+1}^{c,c} =& (A - LC)\tilde{x}_k^{c,c}, &(26c)\\
\tilde{H}_{k+1}^{\tilde{x}} =& \hat{H}_{k+1}^x &(26d)
\end{aligned}$$

with $\tilde{e}_k^c$ and $H_k^{\tilde{e}}$, and $\tilde{x}_k^{c,c}$ and $\tilde{H}_k^{\tilde{x}}$ are the centers and segment matrices of $\tilde{E}_k$ and $\tilde{X}_k^c$, respectively.

Remark 5: Since both $A - HCA - MK_1C$ and $A - LC$ are designed to be Schur matrices, it can be obtained that, for any initial conditions $\tilde{e}_0^c$ and $\tilde{x}_0^{c,c}$, $\tilde{e}_k^c$ and $\tilde{x}_k^{c,c}$ will tend to $\mathbf{0}$ as k approaches ∞. Moreover, for a zonotope, the segment matrix describes its size, while the sizes of residual zonotopes will directly reflect the conservatism of the two FD approaches. Thus, according to (26b) and (26d), if the initial zonotopes are the same, the matrix iterations described by (13b) and (19b) are equivalent to (25a) and (25b), respectively, and at each time instant, the sizes of the corresponding generated zonotopes are also the same. This implies that, since the signals $\tilde{e}_k$ and $\tilde{x}_k^c$ generated by (17f) and (23e) vary around the origin $\mathbf{0}$, they will always be bounded in zonotopes whose sizes are defined by (13b) and (19b) in real-time, respectively.

Under Remark 5, by analyzing (15) and (21), it can be observed that (13b) and (19b) can be used to assess the FD performance (i.e., FD conservatism) of the SUIO-based approach and the IO-based approach, respectively.

B. Comparison of FD Conservatism

According to the results above, the comparison of the two FD methods can be equivalently transformed into comparing $\tilde{H}_k^x$ and H_k^e in (13b) and (19b) (i.e., (25a) and (25b)).

According to Remark 4, regarding the inclusion of the initial conditions, if it is assumed that we use the same initial zonotope for both (25a) and (25b), i.e., $\tilde{E}_0 = \tilde{X}_0^c$, then we can directly use the ultimate sets (i.e., invariant sets) of (25a) and (25b) to compare the FD conservatism of the two approaches due to the fact that $\tilde{E}_k$ and $\tilde{X}_k^c$ will converge to the minimal invariant sets of $\tilde{e}_k^c$ and $\tilde{x}_k^{c,c}$, respectively.

In order to introduce the notion and computation of invariant sets, the results in [3], [4] are recalled.

Theorem 3.1: For a stable plant $x_{k+1} = A_\circ x_k + B_\circ \delta_k$, where $A_\circ$ and $B_\circ$ are constant and $\delta_k \in \Delta = \{\delta : |\delta - \delta^\circ| \leq \bar{\delta}\}$ with δ° and $\bar{\delta}$ constant, letting $A_\circ = V\Lambda V^{-1}$ be the Jordan decomposition of $A_\circ$, the set

$$\begin{aligned}\Phi(\theta) =& \{x \in \mathbb{R}^n : \left|V^{-1}x\right| \leq (I - |\Lambda|)^{-1} \left|V^{-1}B_\circ\right| \bar{\delta} \\ &+ \theta\} \oplus \xi^\circ \end{aligned} \tag{27}$$

is *robust positively invariant* (RPI) and attractive for the system trajectories, with θ any vector with positive components, $\xi^\circ = (I - A_\circ)^{-1} B_\circ \delta^\circ$ and I the identity matrix:

- for any θ, the set $\Phi(\theta)$ is (positively) invariant, that is, if $x_0 \in \Phi(\theta)$, then $x_k \in \Phi(\theta)$ for all $k \geq 0$.
- given $\theta \in \mathbb{R}^n, \theta > 0$, and $x_0 \in \mathbb{R}^n$, there exists $k^* \geq 0$ such that $x_k \in \Phi(\theta)$ for all $k \geq k^*$.

Definition 3.1: The minimal RPI (mRPI) set of a system is defined as an RPI set contained in any closed RPI set and the mRPI set is unique and compact.

Proposition 3.1: ([4]). Considering the dynamics in Theorem 3.1 and denoting X_0 as an initial set, the set sequence

$$X_{j+1} = A_\circ X_j \oplus B_\circ \Delta, \quad j = 0, 1, 2, \ldots,$$

converges to the mRPI set of the dynamics, where if X_0 is an RPI set, such that each iteration of the set sequence is an RPI approximation of the mRPI set.

Remark 6: Since for any initial sets, the set sequences of $\tilde{E}_k$ and $\tilde{X}_k^c$ will converge to their mRPI sets, respectively, the initial condition is not a necessary condition to compare the two methods with their invariant sets.

According to Remark 4, (13b) and (19b) can describe the sizes of zonotopes generated by the set-based dynamics of

$$\tilde{e}_{k+1} = A_{\tilde{e}} \tilde{e}_k + B_{\tilde{e}} \delta_k^{\tilde{e}}, \tag{28a}$$

$$\tilde{x}_{k+1}^c = A_{\tilde{x}} \tilde{x}_k^c + B_{\tilde{x}} \delta_k^{\tilde{x}} \tag{28b}$$

with

$$\begin{aligned} A_{\tilde{e}} =& A - HCA - MK_1C, \\ B_{\tilde{e}} =& [(E_2 - HCE_2) \ -HF \ -MK_1F], \\ \delta_k^{\tilde{e}} =& [\tilde{\omega}_{2,k}^T \ \tilde{\eta}_{k+1}^T \ \tilde{\eta}_k^T]^T, \ \bar{\delta}^{\tilde{e}} = [\bar{\omega}_2^T \ \bar{\eta}^T \ \bar{\eta}^T]^T, \\ A_{\tilde{x}} =& A - LC, \ B_{\tilde{x}} = [-LF \ E], \\ \delta_k^{\tilde{x}} =& [\tilde{\eta}_k^T \ \tilde{\omega}_k^T]^T, \ \bar{\delta}^{\tilde{x}} = [\bar{\eta}^T \ \bar{\omega}^T]^T, \end{aligned}$$

where $\tilde{\omega}_{2,k-1}^T$, $\tilde{\eta}_{k+1}^T$, $\tilde{\eta}_k^T$, $\bar{\omega}_2^T$, $\bar{\eta}^T$ and $\bar{\omega}^T$ are the transposes of $\tilde{\omega}_{2,k-1}$, $\tilde{\eta}_{k+1}$, $\tilde{\eta}_k$, $\bar{\omega}_2$, $\bar{\eta}$ and $\bar{\omega}$, respectively.

Furthermore, we can use one augmented dynamics to describe both (28a) and (28b) together as

$$\tilde{ex}_{k+1} = A_{\tilde{e},\tilde{x}} \tilde{ex}_k + B_{\tilde{e},\tilde{x}} \delta_k^{\tilde{e},\tilde{x}} \tag{30}$$

with

$$A_{\tilde{e},\tilde{x}} = \begin{bmatrix} A_{\tilde{e}} & \mathbf{0} \\ \mathbf{0} & A_{\tilde{x}} \end{bmatrix}, \tilde{ex}_k = \begin{bmatrix} \tilde{e}_k \\ \tilde{x}_k^c \end{bmatrix}, \delta_k^{\tilde{e},\tilde{x}} = \begin{bmatrix} \delta_k^{\tilde{e}} \\ \delta_k^{\tilde{x}} \end{bmatrix}, \tag{31a}$$

$$B_{\tilde{e},\tilde{x}} = \begin{bmatrix} B_{\tilde{e}} & \mathbf{0} \\ \mathbf{0} & B_{\tilde{x}} \end{bmatrix}, \bar{\delta}^{\tilde{e},\tilde{x}} = \begin{bmatrix} \bar{\delta}_k^{\tilde{e}} \\ \bar{\delta}_k^{\tilde{x}} \end{bmatrix}. \tag{31b}$$

Under Theorem 3.1, we can construct an invariant set $\tilde{EX}_{inv}$ for $\tilde{ex}_k$ and further equivalently transform it into

$$\begin{aligned} \tilde{EX}_{inv} =& \{\tilde{ex}_k : -V_{\tilde{e},\tilde{x}}(I - |\Lambda_{\tilde{e},\tilde{x}}|)^{-1} \left|V_{\tilde{e},\tilde{x}}^{-1} B_{\tilde{e},\tilde{x}}\right| \bar{\delta}^{\tilde{e},\tilde{x}} \leq \tilde{ex}_k \\ &\leq V_{\tilde{e},\tilde{x}}(I - |\Lambda_{\tilde{e},\tilde{x}}|)^{-1} \left|V_{\tilde{e},\tilde{x}}^{-1} B_{\tilde{e},\tilde{x}}\right| \bar{\delta}^{\tilde{e},\tilde{x}}\} \end{aligned} \tag{32}$$

with $A_{\tilde{e},\tilde{x}} = V_{\tilde{e},\tilde{x}} \Lambda_{\tilde{e},\tilde{x}} V_{\tilde{e},\tilde{x}}^{-1}$.

Theoretically, we should use the mRPI sets of $\tilde{e}_k$ and $\tilde{x}_k^c$ to compare their FD conservatism. However, in general, it is impossible to obtain the accurate mRPI set for a system. Instead, we will use RPI outer ϵ-estimations of the mRPI sets of $\tilde{e}_k$ and $\tilde{x}_k^c$ to replace their mRPI sets, respectively.

Theorem 3.2: ([4]). For all $\epsilon > 0$, there exists an $s \in \mathbb{N}^+$ such that the following RPI outer ϵ-approximation exists:

$$\Omega \subset \Phi_{s+1} \subset \Omega \oplus \mathbb{B}_p^n(\epsilon), \tag{33}$$

where Ω is the mRPI set, Φ_{s+1} is the $(s+1)$-th iteration from an initial RPI set Φ_0 based on Proposition 3.1, $\mathbb{N}^+$ denotes the set of positive integers and $\mathbb{B}_p^n(\epsilon) = \{x \in \mathbb{R}^n : \parallel x \parallel_p \leq \epsilon\}$, where $\parallel x \parallel_p$ is the p-norm of the vector x.

Thus, under Proposition 3.1 and Theorem 3.2, using $\tilde{EX}_{inv}$ as the initial RPI set of $\tilde{ex}_k$ to iterate the set-based version of (30) $s + 1$ times, we can obtain an RPI outer ϵ-estimation $\tilde{EX}_{inv}^\epsilon$ of the mRPI set of $\tilde{ex}_k$.

Then, according to Theorem 3.2 and (30), by projecting $\tilde{EX}_{inv}^\epsilon$ into the axes of $\tilde{e}_k$ and $\tilde{x}_k^c$, we can obtain the corresponding RPI outer ϵ-estimations of the mRPI sets of $\tilde{e}_k$ and $\tilde{x}_k^c$, respectively, which can be denoted as

$$\tilde{E}_{\epsilon,inv} = \{\tilde{e}_k : -P_{\epsilon,\tilde{e}} \leq \tilde{e}_k \leq P_{\epsilon,\tilde{e}}\}, \quad (34a)$$

$$\tilde{X}^c_{\epsilon,inv} = \{\tilde{x}_k^c : -P_{\epsilon,\tilde{x}} \leq \tilde{x}_k^c \leq P_{\epsilon,\tilde{x}}\} \quad (34b)$$

with $P_{\epsilon,\tilde{e}} \geq \mathbf{0}$ and $P_{\epsilon,\tilde{x}} \geq \mathbf{0}$, where the inequalities are understood as the element-wise form.

By analyzing (34a) and (34b), it can be observed that the bounds for the two RPI sets are given by the vectors $P_{\epsilon,\tilde{e}}$ and $P_{\epsilon,\tilde{x}}$. Thus, regarding the comparison of FD conservatism of the SUIO-based method and the IO-based method, we can have the following conclusions in Proposition 3.2.

Proposition 3.2: For a system (1), under Assumptions 2.1 and 2.2, Proposition 3.1, and Theorems 3.1 and 3.2, with the FD criteria (15) and (21) for the SUIO-based approach, the IO-based approach and the notion of RPI sets, the comparison of FD conservatism of the two approaches can be assessed by the vectors $P_{\epsilon,\tilde{e}}$ and $P_{\epsilon,\tilde{x}}$ in (34) as follows:

- $P_{\epsilon,\tilde{e}} > P_{\epsilon,\tilde{x}}$. The FD conservatism of the SUIO-based method is higher than that of the IO-based method.
- $P_{\epsilon,\tilde{e}} < P_{\epsilon,\tilde{x}}$: The FD conservatism of the SUIO-based method is lower than that of the IO-based method.
- $P_{\epsilon,\tilde{e}} = P_{\epsilon,\tilde{x}}$. In this case, it is considered that the FD conservatism of the SUIO-based method is equal to that of the IO-based Method.
- neither of the previous cases are satisfied. In this case, there simultaneously exist elements in $P_{\epsilon,\tilde{e}}$ are smaller/larger than their counterparts in $P_{\epsilon,\tilde{x}}$. This means that, for some faults, the SUIO-based method is less conservative, while for some other faults, the IO-based method could be better.

Proof : The analysis of these conclusions have been done in detail above. Thus, the proof is omitted here. □

Remark 7: In Proposition 3.2, the inequalities of vectors should be understood element-wise.

IV. Illustrative Example

A numerical example including actuator faults is used as an example to illustrate the proposed approach

$$x_{k+1} = Ax_k + BF_i^a u_k + Ew_k,$$
$$y_k = Cx_k + F\eta_k,$$

with

$$A = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$E = \begin{bmatrix} 0.5 & 0 & 0.3 & 0 \\ 0.2 & 0.4 & 0 & 0.5 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where F_i^a, a diagonal matrix, models actuator faults.

Remark 8: When the system is healthy, F_i^a is the identity matrix. If a fault occurs in an actuator, the corresponding diagonal entry of F_i^a will be a value inside $[0, 1)$.

In order to show the effectiveness of the proposed method, we consider two fault magnitudes in the second actuator:

$$F_1^a = \begin{bmatrix} 1 & 0 \\ 0 & 0.95 \end{bmatrix}, F_2^a = \begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix}.$$

The system has four unknown inputs and for illustrative purposes, we design an SUIO to be insensitive to the first unknown input, which means

$$E^1 = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, E^2 = \begin{bmatrix} 0 & 0.3 & 0 \\ 0.4 & 0 & 0.5 \end{bmatrix}.$$

The parametric matrices of the SUIO are designed as

$$N = \begin{bmatrix} 0.4097 & 0.0179 \\ -0.7594 & -0.1122 \end{bmatrix}, T = \begin{bmatrix} -0.8681 & 1.0851 \\ 1.3194 & -1.6492 \end{bmatrix},$$
$$K_1 = \begin{bmatrix} 0.3167 & 0.3135 \\ 0.6999 & 0.2940 \end{bmatrix}, K_2 = \begin{bmatrix} 0.3993 & 0.0668 \\ -1.8536 & -0.7081 \end{bmatrix},$$
$$K = \begin{bmatrix} 0.7160 & 0.3803 \\ -1.1537 & -0.4141 \end{bmatrix}, M = \begin{bmatrix} 0.6825 & 0.5746 \\ 0.7279 & 0.2220 \end{bmatrix},$$
$$H = \begin{bmatrix} 8.3436 & 4.1409 \\ 3.3899 & 1.5253 \end{bmatrix}.$$

The initial conditions in this simulation are given as

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \hat{X}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \mathbb{B}^2,$$
$$z_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E_0^r = \begin{bmatrix} -0.0739 \\ -0.0296 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbb{B}^2.$$

The gain matrix for the IO is designed as

$$L = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}.$$

We use two sinusoidal inputs to excite the system:

$$u_k^1 = 10\sin(0.2k),\ u_k^2 = 10\sin(0.2k).$$

Additionally, it is assumed that the sets of the process disturbances and measurement noises are bounded by

$$W^1 = \begin{bmatrix} 0 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \mathbb{B}^1, W^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \mathbb{B}^3,$$
$$W = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \mathbb{B}^4.$$

For comparative purposes, two fault scenarios are defined:

- Scenario 1: from $k = 0$ to $k = 20$, the system is healthy, while from $k = 21$ to $k = 50$, the fault F_1^a occurs and
$$V = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \mathbb{B}^2.$$
- Scenario 2: from $k = 0$ to $k = 20$, the system is healthy, while from $k = 21$ to $k = 50$, the fault F_2^a occurs and
$$V = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix} \mathbb{B}^2.$$

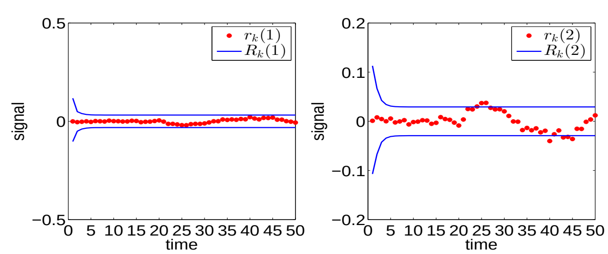

Fig. 1. SUIO-based FD of Scenario 1

According to the results in Section III-B, we consider $\epsilon = 0.001$ and by iterating the set-based version of (30) with an initial RPI set constructed by using Theorem 3.1 by setting $s = 20$, we can compute the RPI outer ϵ-estimations of the mRPI sets of $\tilde{e}_k$ and $\tilde{x}_k^c$. Moreover, for Scenarios 1 and 2, we can further compute $P_{\epsilon,\tilde{e}}$ and $P_{\epsilon,\tilde{x}}$. In this paper, we

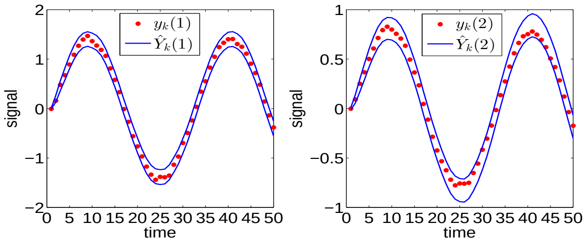

Fig. 2. IO-based FD of Scenario 1

use the superscripts 1 and 2 to denote Scenarios 1 and 2, respectively, and the values of the vectors are given as

$$P^1_{\epsilon,\tilde{e}} = \begin{bmatrix} 0.2216 \\ 0.2380 \end{bmatrix}, P^1_{\epsilon,\tilde{x}} = \begin{bmatrix} 1.4000 \\ 1.0667 \end{bmatrix}, \tag{36a}$$

$$P^2_{\epsilon,\tilde{e}} = \begin{bmatrix} 10.1017 \\ 6.7776 \end{bmatrix}, P^2_{\epsilon,\tilde{x}} = \begin{bmatrix} 8.8250 \\ 1.0667 \end{bmatrix}, \tag{36b}$$

where, it can be observed that

$$P^1_{\epsilon,\tilde{e}} < P^1_{\epsilon,\tilde{x}},\ P^2_{\epsilon,\tilde{e}} > P^2_{\epsilon,\tilde{x}}, \tag{37}$$

which shows that, according to what is presented in Section III-B, for Scenario 1, the FD sensitivity of the SUIO-based method is better, while for for Scenario 2, the IO-based method is better. In order to verify these analysis, simulation results are further shown in Figures 1 and 2, 3 and 4.

For Scenario 1, the results corresponding to the SUIO-based approach and the IO-based FD approach are shown in Figures 1 and 2. It can be observed that the SUIO-based method can detect the actuator fault at time instant $k = 22$, while the IO-based method cannot detect the fault.

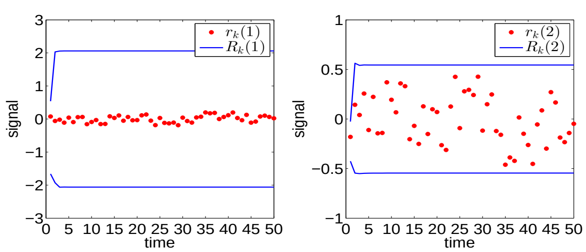

Fig. 3. SUIO-based FD of Scenario 2

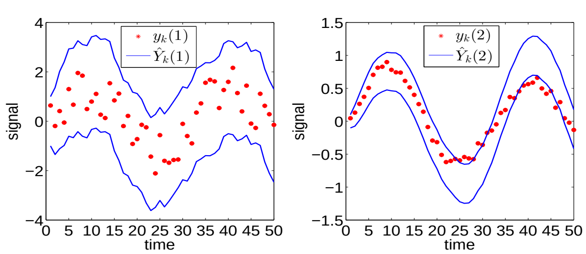

Fig. 4. IO-based FD of Scenario 2

For Scenario 2, the corresponding results are shown in Figures 3 and 4. It can be observed that the SUIO-based method cannot detect the fault, while the IO-based method can detect the fault at time instant $k = 26$.

We can see that the simulation results are completely in agreement with the theoretic results obtained in the paper, which shows that the relative FD conservatism of the two methods is determined by faults, disturbances, noises and the system dynamics, which can be assessed by using the results in Proposition 3.2.

V. Conclusions

This paper focuses on two robust FD approaches and the objective is to analyze the FD performance of both methods in a mathematical way. This paper establishes a comparative framework for the two approaches based on the residual forms (15) and (21) and the notion of invariant sets. It has been obtained that both methods have their advantages in robust FD, which is determined by the faults and the bounds of disturbances and measurement noises. In the future research, we will further study the fourth case *neither of the previous cases are satisfied* introduced in Section III-B.

Acknowledgements

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